Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

Higher Order Linear Differential Equations

Math 240 — Calculus III

Summer 2015, Session II

Tuesday, July 28, 2015



Math 240

Linear DE

- Linear differential operators Familiar stuff Example
- Homogeneous equations

 Linear differential equations of order n Linear differential operators Familiar stuff An example

2. Homogeneous constant-coefficient linear differential equations



Introduction

Higher Order Linear Differential Equations

Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations We now turn our attention to solving **linear differential** equations of order n. The general form of such an equation is $a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x)$, where a_0, a_1, \ldots, a_n , and F are functions defined on an interval I.

The general strategy is to reformulate the above equation as

$$Ly = F$$
,

where L is an appropriate linear transformation. In fact, L will be a *linear differential operator*.



Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

Linear differential operators

Recall that the mapping $D: C^k(I) \to C^{k-1}(I)$ defined by D(f) = f' is a linear transformation. This D is called the **derivative operator.** Higher order derivative operators $D^k: C^k(I) \to C^0(I)$ are defined by composition:

$$D^k = D \circ D^{k-1},$$

so that

$$D^k(f) = \frac{d^k f}{dx^k}.$$

A linear differential operator of order n is a linear combination of derivative operators of order up to n,

$$L = D^{n} + a_1 D^{n-1} + \dots + a_{n-1} D + a_n,$$

defined by

$$Ly = y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y,$$



where the a_i are continous functions of x. L is then a linear transformation $L: C^n(I) \to C^0(I)$. (Why?)

Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

Example

If
$$L = D^2 + 4xD - 3x$$
, then
$$Ly = y'' + 4xy' - 3xy.$$

We have

$$L(\sin x) = -\sin x + 4x\cos x - 3x\sin x, L(x^{2}) = 2 + 8x^{2} - 3x^{3}.$$

Examples

Example If $L = D^2$

f
$$L = D^2 - e^{3x}D$$
, determine
1. $L(2x - 3e^{2x}) = -12e^{2x} - 2e^{3x} + 6e^{5x}$
2. $L(3\sin^2 x) = -3e^{3x}\sin 2x - 6\cos 2x$



Math 240

Linear DE

Linear differential operators Familiar stuff

Example

Homogeneous equations

Homogeneous and nonhomogeneous equations

Consider the general *n*-th order linear differential equation $a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x)$, where $a_0 \neq 0$ and a_0, a_1, \ldots, a_n , and F are functions on an interval I.

If $a_0(x)$ is nonzero on I, then we may divide by it and relabel, obtaining

$$y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_{n-1}(x)y' + a_n(x)y = F(x),$$

which we rewrite as

$$Ly = F(x),$$

where $L = D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n$.

If F(x) is identically zero on I, then the equation is **homogeneous**, otherwise it is **nonhomogeneous**.

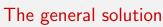


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Linear DE

Linear differential operators Familiar stuff

Homogeneous equations



If we have a homogeneous linear differential equation

Ly = 0,

its solution set will coincide with Ker(L). In particular, the kernel of a linear transformation is a subspace of its domain.

Theorem

The set of solutions to a linear differential equation of order n is a subspace of $C^n(I)$. It is called the **solution space**. The dimension of the solutions space is n.

Being a vector space, the solution space has a basis $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ consisting of n solutions. Any element of the vector space can be written as a linear combination of basis vectors

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x).$$

This expression is called the general solution.



The Wronskian

Higher Order Linear Differential Equations

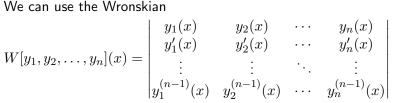
Math 240

Linear DE

Linear differential operators

Familiar stuff Example

Homogeneous equations



to determine whether a set of solutions is linearly independent.

Theorem

Let y_1, y_2, \ldots, y_n be solutions to the *n*-th order differential equation Ly = 0 whose coefficients are continuous on *I*. If $W[y_1, y_2, \ldots, y_n](x) = 0$ at any single point $x \in I$, then $\{y_1, y_2, \ldots, y_n\}$ is linearly dependent.

To summarize, the vanishing or nonvanishing of the Wronskian on an interval *completely characterizes* the linear dependence or independence of a set of solutions to Ly = 0.



The Wronskian

Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

Example

Verify that $y_1(x) = \cos 2x$ and $y_2(x) = 3 - 6 \sin^2 x$ are solutions to the differential equation y'' + 4y = 0 on $(-\infty, \infty)$.

Determine whether they are linearly independent on this interval.

$$W[y_1, y_2](x) = \begin{vmatrix} \cos 2x & 3 - 6\sin^2 x \\ -2\sin 2x & -12\sin x \cos x \end{vmatrix}$$
$$= -6\sin 2x \cos 2x + 6\sin 2x \cos 2x = 0$$
They are linearly dependent. In fact, $3y_1 - y_2 = 0$.



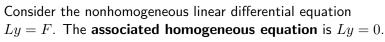
Math 240

Linear DE

Linear differential operators

Familiar stuff Example

Homogeneous equations



Theorem

Suppose $\{y_1, y_2, \ldots, y_n\}$ are *n* linearly independent solutions to the *n*-th order equation Ly = 0 on an interval *I*, and $y = y_p$ is any particular solution to Ly = F on *I*. Then every solution to Ly = F on *I* is of the form

$$y = \underbrace{c_1 y_1 + c_2 y_2 + \dots + c_n y_n}_{y_c} + y_p,$$

for appropriate constants c_1, c_2, \ldots, c_n .

This expression is the **general solution** to Ly = F. The components of the general solution are

- the complementary function, y_c, which is the general solution to the associated homogeneous equation,
- the particular solution, y_p .



Nonhomogeneous equations

Something slightly new

Math 240

Linear DE

Linear differential operators

Familiar stuff

Example

Homogeneous equations

Theorem

If $y = u_p$ and $y = v_p$ are particular solutions to Ly = f(x) and Ly = g(x), respectively, then $y = u_p + v_p$ is a solution to Ly = f(x) + g(x).

Proof.

We have $L(u_p + v_p) = L(u_p) + L(v_p) = f(x) + g(x)$. Q.E.D.



An example

Higher Order Linear Differential Equations

Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

Example

Determine all solutions to the differential equation y'' + y' - 6y = 0 of the form $y(x) = e^{rx}$, where r is a constant.

Substituting $y(x) = e^{rx}$ into the equation yields

$$e^{rx}(r^2 + r - 6) = r^2 e^{rx} + re^{rx} - 6e^{rx} = 0.$$

Since $e^{rx} \neq 0,$ we just need (r+3)(r-2)=0. Hence, the two solutions of this form are

$$y_1(x) = e^{2x}$$
 and $y_2(x) = e^{-3x}$.

Could this be a basis for the solution space? Check linear independence. Yes! The general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-3x}.$$



Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

Example

Determine the general solution to the differential equation

$$y'' + y' - 6y = 8e^{5x}.$$

An example

We know the complementary function,

$$y_c(x) = c_1 e^{2x} + c_2 e^{-3x}$$

For the particular solution, we might guess something of the form $y_p(x) = ce^{5x}$. What should c be? We want $8e^{5x} = y''_p + y'_p - 6y_p = (25c + 5c - 6c)e^{5x}$.

Cancel e^{5x} and then solve 8 = 24c to find $c = \frac{1}{3}$.

The general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-3x} + \frac{1}{3} e^{5x}.$$



Introduction

Higher Order Linear Differential Equations

Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

We just found solutions to the linear differential equation $y^{\prime\prime}+y^{\prime}-6y=0$

of the form $y(x) = e^{rx}$. In fact, we found all solutions. This technique will often work. If $y(x) = e^{rx}$ then

$$y'(x) = re^{rx}, \quad y''(x) = r^2 e^{rx}, \quad \dots, \quad y^{(n)}(x) = r^n e^{rx}.$$

So if $r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n = 0$ then $y(x) = e^{rx}$ is a

solution to the linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$$

Let's develop this approach more rigorously.



Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

Consider the homogeneous linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$$

The auxiliary polynomial

with constant coefficients a_i . Expressed as a linear differential operator, the equation is P(D)y = 0, where

$$P(D) = D^{n} + a_{1}D^{n-1} + \dots + a_{n-1}D + a_{n}.$$

Definition

A linear differential operator with constant coefficients, such as P(D), is called a **polynomial differential operator**. The polynomial

$$P(r) = r^{n} + a_{1}r^{n-1} + \dots + a_{n-1}r + a_{n}$$

is called the **auxiliary polynomial**, and the equation P(r) = 0 the **auxiliary equation**.



The auxiliary polynomial

Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

Example

The equation y'' + y' - 6y = 0 has auxiliary polynomial $P(r) = r^2 + r - 6.$

Examples

Give the auxiliary polynomials for the following equations.

1.
$$y'' + 2y' - 3y = 0$$

2. $(D^2 - 7D + 24)y = 0$
3. $y''' - 2y'' - 4y' + 8y = 0$
 $r^2 - 7r + 24$
 $r^3 - 2r^2 - 4r + 8$

The roots of the auxiliary polynomial will determine the solutions to the differential equation.



Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

Polynomial differential operators commute

The key fact that will allow us to solve constant-coefficient linear differential equations is that polynomial differential operators commute.

Theorem

If P(D) and Q(D) are polynomial differential operators, then $P(D)Q(D)=Q(D)P(D). \label{eq:polynomial}$

Proof.

For our purposes, it will suffice to consider the case where P and Q are linear. $\mathcal{Q.E.D.}$

Commuting polynomial differential operators will allow us to turn a root of the auxiliary polynomial into a solution to the corresponding differential equation.



Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

Linear polynomial differential operators

In our example,

$$y'' + y' - 6y = 0,$$

with auxiliary polynomial

$$P(r) = r^2 + r - 6,$$

the roots of P(r) are r = 2 and r = -3. An equivalent statement is that r - 2 and r + 3 are linear factors of P(r).

The functions $y_1(x)=e^{2x}$ and $y_2(x)=e^{-3x}$ are solutions to $y_1'-2y_1=0$ and $y_2'+3y_2=0,$

respectively.

Theorem

The general solution to the linear differential equation

$$y' - ay = 0$$

is
$$y(x) = ce^{ax}$$

Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

Theorem

Suppose P(D) and Q(D) are polynomial differential operators

$$P(D)y_1 = 0 = Q(D)y_2.$$

If L = P(D)Q(D), then

$$Ly_1 = 0 = Ly_2.$$

Proof.

$$P(D)Q(D)y_2 = P(D)(Q(D)y_2) = P(D)0 = 0$$

$$P(D)Q(D)y_1 = Q(D)P(D)y_1$$

$$= Q(D)(P(D)y_1) = Q(D)0 = 0$$

$$Q.\mathcal{E}.\mathcal{D}.$$

Example

The theorem implies that, since

$$(D-2)y_1 = 0$$
 and $(D+3)y_2 = 0$,



the functions $y_1(x) = e^{2x}$ and $y_2(x) = e^{-3x}$ are solutions to $y'' + y' - 6y = (D^2 + D - 6)y = (D - 2)(D + 3)y = 0.$

Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

Furthermore, solutions produced from different roots of the auxiliary polynomial are independent.

Example

lf

$$y_1(x) = e^{2x}$$
 and $y_2(x) = e^{-3x}$, then
 $W[y_1, y_2](x) = \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix}$
 $= e^{-x} \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5e^{-x} \neq 0.$



Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

Distinct linear factors

If we can factor the auxiliary polynomial into distinct linear factors, then the solutions from each linear factor will combine to form a fundamental set of solutions.

Example

Determine the general solution to y'' - y' - 2y = 0.

The auxiliary polynomial is

$$P(r) = r^{2} - r - 2 = (r - 2)(r + 1).$$

Its roots are $r_1 = 2$ and $r_2 = -1$. The functions $y_1(x) = e^{2x}$ and $y_2(x) = e^{-x}$ satisfy

$$(D-2)y_1 = 0 = (D+1)y_2.$$

Therefore, y_1 and y_2 are solutions to the original equation. Since we have 2 solutions to a 2^{nd} degree equation, they constitute a fundamental set of solutions; the general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-x}.$$



Multiple roots

Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

What can go wrong with this process? The auxiliary polynomial could have a multiple root. In this case, we would get one solution from that root, but not enough to form the general solution. Fortunately, there are more.

Theorem

The differential equation $(D - r)^m y = 0$ has the following m linearly independent solutions:

 $e^{rx}, xe^{rx}, x^2e^{rx}, \dots, x^{m-1}e^{rx}.$

Proof. Check it.

Q.E.D.



Multiple roots

Higher Order Linear Differential Equations

Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

Example

Determine the general solution to y'' + 4y' + 4y = 0.

- 1. The auxiliary polynomial is $r^2 + 4r + 4$.
- 2. It has the multiple root r = -2.
- 3. Therefore, two linearly independent solutions are

$$y_1(x) = e^{-2x}$$
 and $y_2(x) = xe^{-2x}$.

4. The general solution is

$$y(x) = e^{-2x}(c_1 + c_2 x).$$

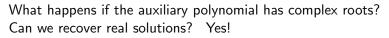


Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations



Complex roots

Theorem

If P(D)y = 0 is a linear differential equation with real constant coefficients and $(D - r)^m$ is a factor of P(D) with r = a + bi and $b \neq 0$, then

- 1. P(D) must also have the factor $(D-\overline{r})^m$,
- 2. this factor contributes the complex solutions $e^{(a\pm bi)x}, xe^{(a\pm bi)x}, \dots, x^{m-1}e^{(a\pm bi)x},$
- 3. the real and imaginary parts of the complex solutions are linearly independent real solutions

 $x^k e^{ax} \cos bx$ and $x^k e^{ax} \sin bx$

for $k = 0, 1, \ldots, m - 1$.



Complex roots

Math 240

Linear DE

Linear differential operators Familiar stuff Example

Homogeneous equations

Example

Determine the general solution to y'' + 6y' + 25y = 0.

- 1. The auxiliary polynomial is $r^2 + 6r + 25$.
- 2. Its has roots $r = -3 \pm 4i$.
- 3. Two independent real-valued solutions are

 $y_1(x) = e^{-3x} \cos 4x$ and $y_2(x) = e^{-3x} \sin 4x$.

4. The general solution is

$$y(x) = e^{-3x}(c_1\cos 4x + c_2\sin 4x).$$

